G_2 COSMOLOGICAL MODELS SEPARABLE IN NON-COMOVING COORDINATES

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Abstract

We study new separable orthogonally transitive abelian G_2 on S_2 models with two mutually orthogonal integrable Killing vector fields. For this purpose we consider separability of the metric functions in a coordinate system in which the velocity vector field of the perfect fluid does not take its canonical form, providing thereby solutions which are non-separable in comoving coordinates in general. Some interesting general features concerning this class of solutions are given. We provide a full classification for these models and present several families of explicit solutions with their properties.

1 Introduction

The study of spatially inhomogeneous cosmological models has been of great interest during past and present decades [1, 2, 3, 4, 5, 6]. The fact that the Universe is not exactly spatially homogeneous and the possibility of obtaining more general solutions for very-early or late universe models are the main reasons for such studies. The spatially inhomogeneous models which have been studied systematically are those space-times admitting a maximally 2-dimensional group of local isometries acting on spacelike surfaces (called G_2 on S_2). Wainwright's classification [3] for the abelian G_2 case is a very useful tool in order to deal with such solutions. This classification contains four sub-cases depending on the features of the two Killing vector fields that generate the group. The most simple sub-case, called B(ii), appears when the Killing vectors are both hypersurface-orthogonal (and hence the group acts orthogonally transitively) and mutually orthogonal. This class with a perfect fluid source has been already treated under some extra conditions, as for example, the imposition of additional homothetic

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or conformal Killing vector fields ([7, 8] and references therein). The most studied simplification has been, however, the assumption of separability of the metric functions in diagonal and canonical form [9, 10, 11].

There have also been general studies on orthogonally transitive G_2 cosmologies from a qualitative point of view, analyzing the autonomous system of first-order partial differential equations coming from the Einstein field equations by using methods from the theory of dynamical sistems [12, 13, 10]. The relations between some of the known explicit solutions and these theoretical studies are also analyzed in [13] and many references therein.

It must be stressed that the majority of the explicitly known solutions [9, 10, 11] have been obtained by means of the separability of the metric functions in *comoving coordinates*, that is to say, such that the velocity vector of the fluid takes its canonical form $\boldsymbol{u} \propto \mathrm{d}x^0$. This involves *two* types of restrictions because the separability of the metric functions is assumed in a *particular* well-defined coordinate system. Thus, solutions which are separable in other *non-comoving* coordinates have not been studied so far. Notice that the choice of comoving coordinates will destroy in general any *previously* assumed separability of the metric functions. By the way, non-comoving coordinates have been already used before (a pioneering paper is [14]).

All this is fully explained in Sections 2 and 4. Section 2 is devoted to presenting some general properties of the orthogonally transitive abelian G_2 models, in particular the so-called "coordinate interchange symmetry" which allows to change the names of the non-ignorable coordinates for general G_2 diagonal perfect-fluid spacetimes. In fact, this property holds also for non-diagonal G_2 models (see [8]). Section 3 is devoted to showing a broadly general property of spatially inhomogeneous perfect-fluid models, that is, that they are in general extendible spacetimes. In other words, solving the Einstein field equations for a perfect fluid one obtains *regions* where the perfect-fluid character of the energy-momentum tensor holds, but these regions may not (and in general they will not) be a complete spacetime because the algebraic type of the energy-momentum tensor cannot hold everywhere. This is a generic property of perfect-fluid inhomogeneous spacetimes, and in particular it holds for the diagonal G_2 cases. This had not been remarked before precisely due to the traditional use of comoving coordinates, because the comoving coordinates hold obviously *only* at the true perfect-fluid region. Actually, the use of non-comoving coordinates, as in the present treatment, provides natural extensions of the incomplete perfect-fluid regions whenever they are extendible. These natural extensions keep the G_2 symmetry of the global spacetime but, of course, the energy-momentum tensor cannot keep its perfect-fluid form. The possible algebraic types allowed for these extensions are also given in section 3.

The separation Ansatz in general coordinates is defined in section 4 in a precise manner. This leads to a classification of the general separable models depending on the values of two natural numbers related to the number of linearly independent functions appearing in the metric. The full classification is presented and analyzed, including the explicit form of the field equations, which now become simple systems of ordinary differential equations. The general kinematical properties of the Weyl tensor, the

Petrov type and other relevant quantities are also given.

Finally, section 5 is devoted to the resolution and the study of the properties of part of the cases classified in section 4. Several solutions are explicitly given and many of them serve as illustrative examples of the method and their general features. In particular, the coordinate interchange symmetry is used to get physically adequate solutions. Furthermore, the power of the method is manifested by presenting one of the solutions in its canonical comoving coordinates, showing that it would have been very difficult indeed to have found it in its comoving form despite the fact that it has a perfectly simple expression in non-comoving coordinates. Some concluding remarks are given at the end of the paper.

2 G_2 cosmological models

We are dealing with orthogonally transitive abelian G_2 models with two mutually orthogonal integrable[1] Killing vectors. The matter content is assumed to be a perfect fluid, thus there exists a time-like vector field $\vec{u}$ (velocity vector) such that the energy-momentum tensor takes the following form:

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}, \qquad u_\alpha u^\alpha = -1, \tag{1}$$

where ρ and p are the energy density and the pressure of the fluid respectively. The existence of two commuting integrable Killing vector fields implies that the velocity vector field is orthogonal to them and invariant under the symmetry group. This, in turn, implies that $\vec{u}$ is integrable. From this last result it can be proven a theorem (see Wainwright [3]) that assures the existence of *local* coordinates $\{x^\alpha\}$, already adapted to both Killings, in which the metric takes a diagonal form and such that $\vec{u} \propto \partial/\partial x^0$ (canonical form of $\vec{u}$), so $\boldsymbol{u} \propto \mathrm{d}x^0$; that is, there always exist coordinates *adapted* simultaneously to the Killings and the velocity vector so that the metric is diagonal.

Nevertheless, we can also choose *non-adapted coordinates* to $\vec{u}$. Of course, this will be relevant only when some restriction has been imposed on the metric functions in a coordinate system. Thus, the assumption of separability allows us to distinguish two cases: the particular case when the coordinates that bring the metric functions in separate form are chosen to be adapted to the velocity vector, and the general one, when these coordinates are not further restricted to be adapted to $\vec{u}$. In this last generalized case, the velocity vector field of the perfect fluid takes its most general form (restricted to be integrable, orthogonal to the Killing vectors and invariant under the G_2 group) in the coordinates that diagonalize the metric and "separate" the metric functions. The line–element for coordinates adapted to both Killing vectors but not necessarily to the velocity vector field reads

$$ds^2 = -F_0^2 \mathrm{d}t^2 + F_1^2 \mathrm{d}x^2 + F_2(F_3^2 \mathrm{d}y^2 + F_3^{-2} \mathrm{d}z^2), \tag{2}$$

[1]By "integrable" vector fields we mean hypersurface orthogonal vector fields.

where $F_\alpha = F_\alpha(t,x)$ and the Killings are $\vec{\xi} = \partial/\partial y$, $\vec{\eta} = \partial/\partial z$, while the velocity 1-form takes its most general form

$$\boldsymbol{u} = u_0(t,x)\boldsymbol{\theta}^0 + u_1(t,x)\boldsymbol{\theta}^1, \tag{3}$$

where $(u_0)^2 - (u_1)^2 = 1$ in the orthonormal co-basis

$$\boldsymbol{\theta}^0 = F_0 \mathrm{d}t, \qquad \boldsymbol{\theta}^1 = F_1 \mathrm{d}x, \qquad \boldsymbol{\theta}^2 = F_2^{1/2} F_3 \mathrm{d}y, \qquad \boldsymbol{\theta}^0 = F_2^{1/2} F_3^{-1} \mathrm{d}z. \tag{4}$$

The non-zero components of the Einstein tensor for the metric (2) in this frame are S_{00}, S_{01}, S_{11}, S_{22} and S_{33}, so the Einstein equations for a perfect fluid become

$$S_{22} = S_{33}\,, \tag{5}$$
$$S_{01}^2 = (S_{00} + S_{22})(S_{11} - S_{22}), \tag{6}$$

where it must be taken into account that the existence of the perfect fluid is restricted to the region defined as follows:

$$2S_{22} + S_{00} - S_{11} \neq 0, \qquad \mathrm{sign}(2S_{22} + S_{00} - S_{11}) = \mathrm{sign}(S_{00} + S_{22}). \tag{7}$$

Note that $\mathrm{sign}(S_{00} + S_{22}) = \mathrm{sign}(S_{11} - S_{22})$ due to (6) whenever $S_{01} \neq 0$. Equations (5) and (6) are the necessary conditions for having two spacelike eigenvectors of the Einstein tensor with the same eigenvalues ($p = S_{22} = S_{33}$) and a third eigenvector with the same eigenvalue (p). Conditions (7) assure both the existence of this third eigenvector as well as its spacelike character. (We refer to section 3 for a deeper discussion on this topic.)

The perfect-fluid quantities defined in region (7) are then

$$p = S_{22}, \qquad \rho = S_{22} + S_{00} - S_{11},$$
$$(u_0)^2 = \frac{S_{00} + S_{22}}{\rho + p}, \qquad (u_1)^2 = \frac{S_{11} - S_{22}}{\rho + p}, \tag{8}$$

where the signs for u_0 and u_1 must be chosen such that the relation

$$S_{01} = (\rho + p) u_0 u_1, \tag{9}$$

which comes from (1) and Einstein's field equations in units with $8\pi G = c = 1$, holds. It still remains, of course, the freedom on the whole sign for $\vec{u}$, which is usually chosen such that u^0 is positive (i.e. u_0 is negative). This means that both $\vec{u}$ and $\partial/\partial x^0$ are future–directed (say).

Let us introduce now an interesting property of coordinate interchange symmetry of the diagonal G_2 geometries which will be very useful for the purposes of classification and the obtaining of solutions in the next section. In fact, this property can be generalized to non-diagonal G_2 B(i) [3] models (see [8]).

Coordinate interchange symmetry

Given any explicit function $f(t,x)$, let us define

$$\tilde{f}(t,x) \equiv f(x,t).$$

Upon this definition, we can construct a new line element (denoted by $\widehat{ds^2}$) which will consist of interchanging the t and x variables in the original metric functions, that is

$$\widehat{ds^2} = \widetilde{g_{\alpha\beta}} \mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

We can compute the Einstein tensor $\widehat{S}_{\alpha\beta}$ of the new metric $\widetilde{g_{\alpha\beta}}$. It is then easy to find the relation between $\widehat{S}_{\alpha\beta}$ and $S_{\alpha\beta}$, which is simply:

$$\begin{array}{lll} \widehat{S}_{22} = -\widetilde{S_{22}}, & \widehat{S}_{33} = -\widetilde{S_{33}}, & \\ \widehat{S}_{01} = \widetilde{S_{01}}, & \widehat{S}_{00} = \widetilde{S_{11}}, & \widehat{S}_{11} = \widetilde{S_{00}}. \end{array}$$

The first consequence of these relations is that the Einstein equations (5) and (6) are invariant under this coordinate interchange in the sense that the Einstein equations for the new line-element are the old equations (5) and (6) with t and x interchanged in the metric functions and, of course, in the derivative operators (changing dots and primes). In other words and explicitly

$$\begin{array}{rcl} \widehat{S}_{22} - \widehat{S}_{33} = 0 & \Longleftrightarrow & \widetilde{S_{22}} - \widetilde{S_{33}} = 0, \\ (\widehat{S}_{00} + \widehat{S}_{22})(\widehat{S}_{11} - \widehat{S}_{22}) = \widehat{S}_{01}{}^2 & \Longleftrightarrow & (\widetilde{S_{00}} + \widetilde{S_{22}})(\widetilde{S_{11}} - \widetilde{S_{22}}) = \widetilde{S_{01}}{}^2. \end{array}$$

In short, this means that if ds^2 is solution of equations (5) and (6), so is $\widehat{ds^2}$. However, notice that the region of existence of the perfect fluid for $\widehat{ds^2}$, given by

$$2\widehat{S}_{22} + \widehat{S}_{00} - \widehat{S}_{11} \neq 0, \qquad \mathrm{sign}(2\widehat{S}_{22} + \widehat{S}_{00} - \widehat{S}_{11}) = \mathrm{sign}(\widehat{S}_{00} + \widehat{S}_{22}), \tag{10}$$

does not coincide in general neither with the previous region (7) nor with the "tilded" region (7) (with t and x interchanged). This region must be found separately in each particular case. In fact, the *only* thing that can be said in general is that the region where $2\widehat{S}_{22} + \widehat{S}_{00} - \widehat{S}_{11} = 0$ is the region where $2S_{22} + S_{00} - S_{11} = 0$ with t and x interchanged because

$$2\widehat{S}_{22} + \widehat{S}_{00} - \widehat{S}_{11} = -(2\widetilde{S_{22}} + \widetilde{S_{00}} - \widetilde{S_{11}}).$$

With regard to the fluid quantities, the coordinate interchange transformation induces the following changes

$$(\widehat{u}_0)^2 = -\frac{\widetilde{S_{11}} - \widetilde{S_{22}}}{\widetilde{\rho} + \widetilde{p}} \left(= \text{“} - (\widetilde{u^1})^2\text{”}\right), \qquad (\widehat{u}^1)^2 = -\frac{\widetilde{S_{00}} + \widetilde{S_{22}}}{\widetilde{\rho} + \widetilde{p}} \left(= \text{“} - (\widetilde{u^0})^2\text{”}\right), \tag{11}$$

$$\widehat{\rho} = -\widetilde{\rho}, \qquad \widehat{p} = -\widetilde{p}.$$

Note that in (11) there is no contradiction in the signs of the quoted expressions because they stand for their expressions (without the minus sign) in (8), which are positive in the region (7), but change to be negative when they get transformed and defined in the new region (10).

3 A broadly general property of the G_2 solutions[2]

We will assume that both Killing vector fields are globally defined (well defined over the whole space-time manifold). If this did not happen, we could take the open subset of the original manifold admitting two spacelike isometries as the manifold itself. Notice that (7) involves t and x in general, so that (7) restricts the allowed values of the coordinates (apart from very particular cases which involve only the parametres of the solution). Then, formulae (7) show that there appear different regions for G_2 on S_2 spacetimes depending on the algebraic type of the Einstein tensor at their points. Actually, this is a general feature as will be shown in what follows.

Equations (5) and (6) restrict the Einstein tensor, and hence the energy-momentum tensor via the Einstein equations, to have four possible algebraic types: {1,(111)}, {(1,11)1}, {(1,111)}, and {(2,11)} in Segré's notation (see [15] and references therein). Only the first three types admit a timelike eigenvector (which is unique only in the first case), while in the fourth type there exists a null eigendirection. The explicit canonical form of $S_{\alpha\beta}$ for each Segré type together with the regions where the respective types hold are given below:

a) $S_{\alpha\beta} = \rho\, v^0_\alpha v^0_\beta + p\,(v^1_\alpha v^1_\beta + v^2_\alpha v^2_\beta + v^3_\alpha v^3_\beta)$, at the region defined by

$$\{2S_{22} + S_{00} - S_{11} \neq 0,\ \mathrm{sign}(2S_{22} + S_{00} - S_{11}) = \mathrm{sign}(S_{00} + S_{22})\},$$

b) $S_{\alpha\beta} = -a\, w^0_\alpha w^0_\beta + b\, w^1_\alpha w^1_\beta + a\,(w^2_\alpha w^2_\beta + w^3_\alpha w^3_\beta)$, at the region given by

$$\{2S_{22} + S_{00} - S_{11} \neq 0,\ \mathrm{sign}(2S_{22} + S_{00} - S_{11}) = -\mathrm{sign}(S_{00} + S_{22})\},$$

ci) $S_{\alpha\beta} \propto g_{\alpha\beta}$, at the region with $\{2S_{22} + S_{00} - S_{11} = 0,\ S_{01} = 0\}$,

cii) $S_{\alpha\beta} = c\, k_\alpha k_\beta - d\, g_{\alpha\beta}$, $\quad (k_\alpha k^\alpha = 0,\ c \neq 0)$, at the region

$$\{2S_{22} + S_{00} - S_{11} = 0,\ S_{01} \neq 0\},$$

where both $\{\boldsymbol{v}^\alpha\}$ and $\{\boldsymbol{w}^\alpha\}$ are orthonormal cobases. The case a) corresponds to the perfect fluid region which we will call the $\mathcal{A}$-region from now on. The zone where the case b) holds will be called the $\mathcal{B}$-region. Here, there is a *timelike* eigenvector of $S_{\alpha\beta}$ with eigenvalue equal to $p \equiv S_{22} = S_{33}$ (in $\mathcal{A}$, all three eigenvectors with eigenvalue p are spacelike). The region defined by $2S_{22} + S_{00} - S_{11} = 0$ is the border $\mathcal{F}$ (cases ci) and cii)), that divides the space-time manifold in the $\mathcal{A}$ and $\mathcal{B}$ regions in the sense that for every continuous curve containing points in $\mathcal{A}$ and $\mathcal{B}$, there always exists at least one point of the curve in $\mathcal{F}$ (this is why we call it a border; see also [16]). The assumption of analyticity of the Einstein tensor on the whole manifold assures that $\mathcal{F}$

[2] We have entitled this section referring to the G_2 inhomogeneous models which are being treated in the present work, but this property or similar ones apply to more general situations.

is either the entire spacetime or its interior is empty in the manifold topology, while the mere assumption of smoothness implies that the $\mathcal{A}$ and $\mathcal{B}$-regions are open sets.

The behaviour of the fluid velocity vector (defined in $\mathcal{A}$) when approaching the border $\mathcal{F}$ varies depending on whether case ci) or cii) holds at $\mathcal{F}$ (this can be easily seen from (8) and (6), or also from (9)): if $S_{01} \neq 0$ at $\mathcal{F}$, the components of $\vec{u}$ in the orthonormal co-basis must diverge when approaching $\mathcal{F}$, but this does not necessarily happen when $S_{01} = 0$ at points of $\mathcal{F}$. Then, in a general situation there will be parts of the border $\mathcal{F}$ where $\vec{u}$ diverges (see [17] for a general discussion).

Therefore, in general, a space-time corresponding to a solution of equations (5) and (6) will be divided into three regions (not necessarily connected) depending on the Segré type of the energy-momentum tensor at their points. This could be somewhat expected due to the quite general conditions we are using. Our main interest, of course, will be the perfect-fluid region $\mathcal{A}$, where we can construct a coordinate system in which $\vec{u}$ takes its canonical (comoving) form while keeping the diagonal form of the metric (see the previous section 2). These coordinates are only defined in this region in general, because the change of coordinates is not valid where $\vec{u}$ diverges, that is, on $\mathcal{F}$, so the search of perfect-fluid solutions in comoving coordinates may lead to solutions with coordinate singularities that would correspond to an $\mathcal{F}$ border.

Therefore, it seems a natural feature of the G_2 perfect-fluid solutions to be extendible across $\mathcal{F}$, and the extensions cannot keep the perfect-fluid character over the whole spacetime. Of course, the above problem has a solution if we do not use comoving coordinates, which is our purpose in this paper. By using non-comoving coordinates and solving Einstein's equations (5) and (6) we obtain, in one single stroke, both regions $\mathcal{A}$ and $\mathcal{B}$ and the border $\mathcal{F}$. Thus, by using this method of obtaining perfect-fluid solutions we also get extensions of the $\mathcal{A}$ regions, that is, of the perfect-fluid regions which are extendible across the border $\mathcal{F}$ where the (comoving) coordinate singularity appears.

4 Separability in general: A classification for the general models

Due to the diagonal form of the metric (2), the notion of separability will be applied to the metric functions in the obvious way:

$$F_\alpha(t,x) = \mathcal{T}_\alpha(t)\mathcal{X}_\alpha(x).$$

Once this is assumed, two different cases appear: (i) t,x adapted to $\vec{u}$ (comoving coordinates) and (ii) t,x non-adapted to $\vec{u}$ (non-comoving coordinates).

All the possible solutions in case (i) have been already identified and studied in [9, 10, 11]. Ref. [9] finds the general solution under the extra assumption $F_3 = F_3(t)$ (see (2)) which implies that the three-slices orthogonal to the fluid congruence are conformaly flat. In ref.[10], the remaining solutions are identified unless in the special

case of $p = \rho$. Finally, [11], provides the $p = \rho$ solutions which complete the comoving case. In this paper, we study the second (non-comoving) case.

With the help of separability we get the following line element

$$ds^2 = e^{2f_1}(-\mathrm{d}t^2 + \mathrm{d}x^2) + e^{f_2}(e^{2f_3}\mathrm{d}y^2 + e^{-2f_3}\mathrm{d}z^2), \tag{12}$$

where $f_a(t,x) = T_a(t) + X_a(x)$ $(a = 1,2,3)$, while $\boldsymbol{u}$ takes its most general (non-canonical) form (3). The explicit components of the Einstein tensor in the orthogonal frame (4) are given in Appendix A. Equations (5) and (6) read then

$$\ddot{T}_3 + \dot{T}_2\dot{T}_3 = X_3'' + X_2'X_3' = \mathrm{K}, \tag{13}$$

$$\left(\dot{T}_1X_2' + \dot{T}_2\left(X_1' - \frac{1}{2}X_2'\right) - 2\dot{T}_3X_3'\right)^2 = (M_7(t) + N_8(x))(M_8(t) + N_7(x)), \tag{14}$$

where the dots and primes denote derivatives with respect to t and x respectively, K is a separation constant, and the $M(t)$'s and $N(x)$'s functions stand for definite combinations of the first and second derivatives of the metric functions which are explicitly defined in Appendix A. Equation (14) can be re-written in the form (see Appendix A)

$$\sum_{i=1}^{8} M_i(t)N_i(x) + M_7(t)M_8(t) + N_7(x)N_8(x) = 0, \tag{15}$$

and differentiating this last expression with respect to t and x we get the following equation:

$$\sum_{i=1}^{8} \dot{M}_i N_i' = 0. \tag{16}$$

This implies that if we define n and q to be the number of linearly independent functions among the sets $\{\dot{M}_i\}$ and $\{N_i'\}$ respectively, we have $q \leq 8 - n$, that is, there are, at most, $8 - n$ linearly independent functions among the $\{N_i'\}$. Thus, at first sight, n might take all values from 0 to 8, but this *will not be necessary* eventually because of the property of coordinate interchange symmetry we have shown previously in section 2: thanks to the $t \leftrightarrow x$ "symmetry", we have a way to relate solutions of equations (13) and (14) interchanging their n and q numbers. Thus, if a solution has $n = 3$ and $q = 5$, for instance, then by using the coordinate interchange we get another solution with $q = 3$ and $n = 5$, and so on. Therefore we must only treat the Einstein equations for the values of n ranging from 0 to 4. The rest of cases can be studied by means of the coordinate interchange.

Notice however that, in fact, we have only three original functions of t, $\{T_a\}$. Therefore, we can define another integer m as the number of linearly independent functions among the $\{T_a\}$ (obviously m runs from 1 to 3 because $m = 0$ avoids any dependence on t of the metric). Of course, m will be related with the previously defined n. The combination of both n and m gives us a way of obtaining solutions and a classification for them.

To proceed with this classification we will start with any value of m, and then this m is related on each case (using some results given in the Appendix B) with the number of linearly independent functions among the set $\{1, M_i\}$, which is exactly $n+1$ due to Lemmas 1 and 2 of Appendix B. Then, each of the three cases $m = 1, 2, 3$ will be divided into the possible values that n can take for each case. For a given pair $\{m, n\}$ some subdivisions may appear depending on the posible relations between the set of functions $\{1, M_i\}$. These different possibilities will be given below. Nevertheless, at this stage, we have not used yet the equations (13) and (14) explicitly, from where new relations arise. These new relations together with the previous ones provide the systems of differential equations for the t-functions and for the x-functions, from where new restrictions may appear after their compatibilization (if necessary). Therefore, the aim of the next subsections is to give the steps in dividing the cases for a given m (subsections 4.1, 4.2 and 4.3). Then, in subsetion 4.4, equations (13) and (14) are treated in order to present the complete set of equations for the t- and $x-$functions. All throughout the next subsections, the latin lower-case characters will stand for constants.

4.1 Case $m = 1$

We have for this case $T_a(t) = c_a T(t)$ and we need to impose $T(t) \not\equiv 0$ and at least a non-vanishing c_a to assure one linearly independent function among the $\{T_a\}$. Furthermore, $\dot{T}$ will be non-zero because otherwise the metric would be static. It is very easy now to establish the possible values of n because, for L running from 1 to 6, we have $M_L \propto \dot{T}^2$, and the only other possible independent function contained in $\{M_i\}$ is $\ddot{T}$, appearing in M_7 and M_8. Therefore there are at most 3 linearly independent functions involved in $\{1, M_i\}$ and, consequently, n can take the values 0,1, and 2, dividing the case $m = 1$ into three subclasses. We give now the characterization of these subclasses for a given n that will be used for the full analisys of the case $m = 1$ in Section 5.

I. $n = 0$ **:** this means $\dot{M}_i = 0$ for $(i = 1 \ldots 8)$, which is equivalent to $\dot{T} = 1$ (the constants are absorbed by the c_a coefficients).

II. $n = 1$ **:** to avoid the previous case the two linearly independent functions must be chosen as $\{1, \dot{T}^2\}$ and we must also have $M_7 = c_{71}\dot{T}^2 + b_7$ and $M_8 = c_{81}\dot{T}^2 + b_8$. Nevertheless, b_7 and b_8 can be absorbed by M_7 and M_8 (see Appendix A). This case, taking into account the t-equation in (13) and the fact that there is at least a non-vanishing c_a, implies an equation for $T(t)$ with the following form: $\ddot{T} = a\dot{T}^2 + b$.

III. $n = 2$ **:** now, there is a relation of the form $a + b\dot{T}^2 + cM_7 + dM_8 = 0$ and there appear two different possibilities depending on wether $c \neq 0$ or $c = 0$.

- **(i)** $c \neq 0$. Then we can take $\{1, \dot{T}^2, M_8\}$ as linearly independent functions, and $M_7 = c_{71}\dot{T}^2 + c_{72}M_8$.

(ii) $c = 0 (\Rightarrow d \neq 0)$ Then $\{1, \dot{T}^2, M_7\}$ can be chosen as linearly independent functions, and $M_8 = c_{81}\dot{T}^2$.

Again, the possible additional constants have been absorbed in M_7 and M_8 as in the previous case.

4.2 Case $m = 2$

In this case we have $T_a(t) = c_a T(t) + d_a K(t)$ where $\{T(t), K(t)\}$ are two linearly independent functions. We also need to impose, of course, that the matrix composed by c_a's and d_a's has rank two. This case can also be treated dealing directly with the $\{T_a\}$ functions and dividing this class depending of which pair is taken to be linearly independent, but we have preferred to introduce the functions $T(t)$ and $K(t)$ for the sake of compactness and brevity.

In fact, $1, T(t), K(t)$ can be assumed to be three linearly independent functions, as otherwise we could reduce this class to the case $m = 1$. For, suppose, on the contrary, that there existed a linear relation $aT(t) + bK(t) + c = 0$. Then, if $a = 0$ we would have that $K(t)$ is a constant and therefore, it could be set equal to zero because the terms $e^{d_a K}$ in the metric can be absorbed into the coordinates. If $a \neq 0$ we would have $T(t) = -(b/a)K(t) - c/a$, thus $T_a(t) = (-c_a b/a + d_a)K(t) - c_a c/a$, and again, redefining the constants and absorbing the constant terms into the coordinates, we could set $T(t) \equiv 0$, in contradiction. From Lemma 1 it follows then that $\dot{T}$ and $\dot{K}$ are two linearly independent functions, and therefore, Lemma 3 of the Appendix B implies that the set $\{\dot{T}^2, \dot{K}^2, \dot{T}\dot{K}\}$ consists of 3 linearly independent functions. This, in turn, means that among $\{M_L\}$ $(L = 1 \ldots 6)$ there are exactly 3 linearly independent functions (see Appendix B).

In M_7 and M_8 there appear two other functions ($\ddot{T}$ and $\ddot{K}$) that can be linearly independent from the rest. Therefore, in the set $\{1, M_i\}$ we have a minimum of 3 linearly independent functions and a maximum of 6 so that n can take the values 2,3,4, and 5, but this last case ($n = 5$) does not need to be treated thanks to the coordinate interchange symmetry $t \leftrightarrow x$. In summary, the case $m = 2$ is divided into three subclasses $n = 2$, $n = 3$ and $n = 4$.

4.3 Case $m = 3$

In this case the three functions $\{T_a\}$ are linearly independent. Nevertheless, we will use here three generic independent functions in order to give a more compact presentation of the cases. Thus, let $T(t), K(t), Q(t)$ be three linearly independent functions such that $T_a(t) = c_a T(t) + d_a K(t) + e_a Q(t)$, where the determinant of the 3x3 matrix composed by the constants must be non-zero.

Analogously to what is explained in the second paragraph of subsection 4.2, the set $\{1, T(t), K(t), Q(t)\}$ consists of four linearly independent functions, as otherwise we could reduce this class to the case $m = 2$. From Lemma 1 it follows then that $\dot{T}$, $\dot{K}$,

and $\dot{Q}$ are three linearly independent functions, and Theorem 1 implies then that there are at least 5 linearly independent functions among the set $\{\dot{T}^2, \dot{K}^2, \dot{Q}^2, \dot{T}\dot{K}, \dot{T}\dot{Q}, \dot{K}\dot{Q}\}$, that is, among the M_L functions, and hence, among $\{1, M_i\}$. Therefore, n can take the values from 4 to 8, so that only the case $n = 4$ has to be treated.

The three cases with $m = 2$ and the single case with $m = 3$ will not be solved explicitly in this paper. Nevertheless, we present the whole set of equations and the kinematical quantities *in general* in the next two subsections.

4.4 The complete set of equations

Now, we proceed with the obtention of the complete set of equations coming from equations (13) and (14) once we have chosen the pair $\{m, n\}$. As was explained above, these equations together with the systems given in the previous subsections will form the full set of equations for the t- and x- functions. In fact, equations (13) need no further treatment and will not be repeated in this subsection. Thus, we focus on equation (14) written in its form (15).

We choose $\{1, m_A(t)\}\,(A = 1 \dots n)$ to be the $n + 1$ given linearly independent functions such that

$$M_i(t) = \sum_{A=1}^{n} c_{iA} m_A(t) + b_i \qquad (i = 1 \dots 8), \tag{17}$$

where the constants c_{iA} form a matrix of rang n.[3] Therefore, we take the functions $m_A(t)$ and the constants b_i to be the fundamental objects with regard to the M_i functions. Now, from (16), using (17) and the fact that $\{\dot{m}_A(t)\}$ are lineraly independent (Lemma 1), there appear n linearly independent relations between N'_i. This implies that there are at most $8 - n$ linearly independent functions in $\{N'_i\}$. Thus, we choose $\{1, n_B(x)\}\,(B = 1 \dots (8 - n))$ to be the $9 - n \equiv (8 - n) + 1$ linearly independent functions such that

$$N_i(x) = \sum_{B=1}^{8-n} d_{iB} n_B(x) + a_i \qquad (i = 1 \dots 8), \tag{18}$$

where the constants d_{iB} have no restriction a priori.

Now, derivating equation (15) with respect to x and using (17) we get

$$\sum_{A=1}^{n} \left(\sum_{i=1}^{8} c_{iA} N'_i \right) m_A(t) + \sum_{i=1}^{8} b_i N'_i + (N'_7 N'_8)' = 0.$$

Using the fact that $\{1, m_A(t)\}$ are linearly independent functions, from this last expression it follows that $\sum_i^8 c_{iA} N'_i = 0$ and $\sum_i^8 b_i N'_i + (N_7 N_8)' = 0$. Integrating this pair

[3]Note that we could set $b_7 = b_8 = 0$ without loss of generality (see Appendix A), but this will not be always used.

of expressions we get

$$\sum_{i=1}^{8} c_{iA} N_i = \mathcal{C}_A \qquad (A = 1 \ldots n), \tag{19}$$

$$\sum_{i=1}^{8} b_i N_i + N_7 N_8 = B, \tag{20}$$

where $\mathcal{C}_\mathcal{A}$ and B are constants. Differentiating equation (14) with respect to t and using (18) we obtain the corresponding expressions for the M_i functions. The equation analogous to (19) does not give relevant information (only relations between constants that will not be used at the end), while the analogous to (20) reads

$$\sum_{i=1}^{8} a_i M_i + M_7 M_8 = A. \tag{21}$$

We take now the original Eq.(15) and substitute the functions M_i from (17) and the terms $M_7 M_8$ and $N_7 N_8$ isolated from (21) and (20) respectively. Using again the fact that $\{1, m_A(t)\}$ are lineraly independent and taking (19) into account, the following relations arise

$$\sum_{i=1}^{8} a_i \, b_i = A + B, \qquad \sum_{i=1}^{8} a_i \, c_{iA} = \mathcal{C}_A.$$

Putting these last expressions into (21) we finally get

$$\sum_{A=1}^{n} \mathcal{C}_A m_A + M_7 M_8 + B = 0. \tag{22}$$

Thus, equation (15) splits into the equivalent set of *ordinary* differential equations (19), (20), and (22). Therefore, the complete set of equations consists of (13),(19),(20), and (22) together with the relations between the t-functions arising from the classification of the previous subsections.

4.5 Kinematical quantities and the Weyl tensor

For the sake of compactness in the expressions of the kinematical properties of $\vec{u}$ we define first the following objects

$$\begin{aligned} w_0 &\equiv e^{2f_1} (S_{00} + S_{22}) = M_7(t) + N_8(x), \\ w_1 &\equiv e^{2f_1} (S_{11} - S_{22}) = M_8(t) + N_7(x), \\ \Sigma &\equiv e^{2f_1} S_{01}, \end{aligned}$$

so the Einstein equation (6) reads simply $w_0 w_1 = \Sigma^2$, and the perfect-fluid quantities take then the following form

$$\rho + p = e^{-2f_1}(w_0 - w_1), \qquad \rho - p = e^{-2f_1}(\ddot{T}_2 + \dot{T}_2^{\,2} - X_2'' - X_2'^2),$$
$$(u_0)^2 = \frac{w_0}{w_0 - w_1}, \qquad (u_1)^2 = \frac{w_1}{w_0 - w_1}.$$

The $\mathcal{A}$-region defined in section 3 is given now by the conditions $w_0 - w_1 \neq 0$, $\mathrm{sign}(w_0 - w_1) = \mathrm{sign}(w_0)$, which can be combined with the extra requirement $\rho + p > 0$ to give

$$w_0 > 0, \qquad\qquad w_0 - w_1 > 0, \tag{23}$$

which become the necessary and sufficient conditions to have a perfect-fluid source satisfying the energy condition $\rho + p > 0$. From now on, the domain defined by these conditions (23) will be referred to as the $\mathcal{A}_E$-region. The relative signs of u_0 and u_1 are determined then by $\Sigma = (w_0 - w_1)u_0 u_1$.

The fact that $\vec{u}$ is integrable implies that its vorticity vanishes, that is, $\omega_{\alpha\beta} = 0$. The expansion reads as follows,

$$\theta = \frac{e^{-f_1}}{u_0(w_0-w_1)^2}\left\{(w_0-w_1)\left[\Sigma\left(X_1' + X_2'\right) - w_0\left(\dot{T}_1 + \dot{T}_2\right)\right]\right.$$
$$\left. + w_0\Sigma' - w_0'\Sigma + \frac{1}{2}(\dot{w}_0 w_1 - w_0 \dot{w}_1)\right\}.$$

The non-zero components of the acceleration computed in the co-basis given in (4) are

$$\mathrm{a}_0 = \frac{u_1}{u_0}\mathrm{a}_1,$$
$$\mathrm{a}_1 = -\frac{e^{-f_1}}{(w_0-w_1)^2}\left\{(w_0-w_1)\left(\Sigma\dot{T}_1 - w_0 X_1'\right)\right.$$
$$\left. + w_0\dot{\Sigma} - \dot{w}_0\Sigma + \frac{1}{2}(w_0' w_1 - w_0 w_1')\right\},$$

while the shear tensor has the following non-zero components

$$\sigma_{00} = \left(\frac{u_1}{u_0}\right)^2\sigma_{11},$$
$$\sigma_{01} = \frac{u_1}{u_0}\sigma_{11},$$
$$\sigma_{11} = -e^{-f_1}\frac{u_0}{(w_0-w_1)}\left(\Sigma X_2' - w_0\dot{T}_2\right) + \frac{2}{3}(u_0)^2\theta,$$
$$\sigma_{22} = -\frac{e^{-f_1}}{u_0(w_0-w_1)}\left(w_0\dot{T}_3 - \Sigma X_3'\right) + \frac{1}{2}\frac{1}{(u_0)^2}\sigma_{11},$$
$$\sigma_{33} = \frac{e^{-f_1}}{u_0(w_0-w_1)}\left(w_0\dot{T}_3 - \Sigma X_3'\right) + \frac{1}{2}\frac{1}{(u_0)^2}\sigma_{11},$$

and the shear scalar $2\sigma^2 \equiv \sigma_{\alpha\beta}\sigma^{\alpha\beta}$ is then,

$$\sigma^2 = \frac{3}{4}\left[\frac{2}{3}\theta - e^{-f_1}\frac{u_0}{(w_0-w_1)}\left(\Sigma X_2' - w_0\dot{T}_2\right)\right]^2 + \left[\frac{e^{-f_1}}{u_0(w_0-w_1)}\left(w_0\dot{T}_3 - \Sigma X_3'\right)\right]^2.$$

The non-vanishing scalars of the Weyl tensor computed in the null tetrad $\mathbf{k} = 2^{-1/2}(\boldsymbol{\theta}^0 - \boldsymbol{\theta}^1)$, $\boldsymbol{l} = 2^{-1/2}(\boldsymbol{\theta}^0 + \boldsymbol{\theta}^1)$, $\mathbf{m} = 2^{-1/2}(\boldsymbol{\theta}^2 + i\,\boldsymbol{\theta}^3)$, where $\boldsymbol{\theta}^{(\alpha)}$ are given in (4), are [15]

$$\Psi_0 + \Psi_4 = 2e^{-2f_1}\left[\dot{T}_1\dot{T}_3 + X_1'X_3' - A\right], \tag{24}$$

$$\Psi_0 - \Psi_4 = e^{-2f_1}\left[2\left(\dot{T}_1X_3' + X_1'\dot{T}_3\right) - \dot{T}_3X_2' - X_3'\dot{T}_2\right], \tag{25}$$

$$\Psi_2 = \frac{1}{12}e^{-2f_1}\left[4\left(\dot{T}_3^{\,2} - X_3'^2\right) + 2\left(X_1'' - \ddot{T}_1\right) + \ddot{T}_2 - X_2''\right], \tag{26}$$

where we have only used equations (13) from which we have isolated both $\ddot{T}_3$ and X_3'' (which appear in Ψ_0 and Ψ_4). From (24)-(26) it follows that the solutions will be in general (and at generic points) of Petrov type I.

5 Full analysis of the case $m = 1$ and explicit solutions

In this section we treat the case $m = 1$ given in subsection 4.1 so that $T_a(t) = c_aT(t)$. We give the equations for each of the subcases taking into account the results of subsection 4.4 and then some particular examples will be solved. Unless otherwise is stated, throughout this section the generic term "solutions" will stand for maximally G_2 not included in the previous works on separable comoving coordinates [9, 10, 11], that is, they will non-separable in comoving coordinates "a priori". Furthermore, a relationship between n and the type of equation of state appears.

5.1 $n = 0$

As was shown in subsection 4.2, we have now $\dot{T}_a(t) = c_a$, so that from (42) and (44) of the Appendix A and (17) we get the values for $b_i = \{c_1^2,\, c_2^2,\, c_3^2,\, c_1c_2,\, c_1c_3,\, c_2c_3,\, c_1c_2 - 2c_3^2,\, c_2(c_1 - 1/2c_2)\}$. Note that we have not used the fact that one can have $b_7 = b_8 = 0$, so we set $\alpha = \beta = 0$ in (44) and (45).

Equations (13) become simply $\mathrm{K} = c_2c_3$ and

$$X_3'' + X_2'X_3' = c_2c_3, \tag{27}$$

while Eqs. (19) provide no relations, (22) fixes B and finally (20) reads

$$\sum_{i=1}^{8} b_iN_i + N_7N_8 = -c_2\left(c_1 - \frac{1}{2}c_2\right)(c_1c_2 - 2c_3^2), \tag{28}$$

where N_i are defined in Appendix A. Equations (27) and (28) form a system of two first-order ordinary differential equations for the three unknown functions X_a'. Thus, in general the solutions depend on an arbitrary function, allowing for several further Ansatzs in order to find explicit solutions of this system of equations. In particular,

this freedom can be used in principle to demand some extra property of the solutions, such as particular equations of state. Indeed, it is possible to find solutions with a $p = \gamma\rho$ equation of state including $\gamma = 0$, that is, dust models. In fact, $X_a(x) = d_a X(x) + l_a x$ such that $X''(x) \neq 0$ and $p = 0$ leads to a family of dust solutions that can be generalized to give a bigger family of algebraically general dust G_2 models [18]. Nevertheless, these solutions have $q = 3$ (number of linearly independent functions among $\{N_i'\}$) so that they will appear, within our scheme, when $n = 3$ in the $m = 2$-case after a $t \leftrightarrow x$ change. In this way, in order to avoid any superposition of solutions between the different subcases (using the $t \leftrightarrow x$, $n \leftrightarrow q$ "symmetry") and to keep a 'coherent' classification, we should look only for solutions with the following values of $q \in \{0, 5, 6, 7, 8\}$. The rest of the cases ($q = 1, 2, 3, 4$) can be left to the study of cases ($n = 1, 2, 3, 4,\ q = 0$). The cases with $q \geq 5$, following the reasoning given in Section 4, need at least two linearly independent functions among $\{X_a(x)\}$. The solutions depend on the extra assumption which closes the system (27)-(28).

In the remaining possibility $q = 0$ we have that $X_a'(x) = k_a$, where k_a are constants. These spacetimes always admit a third Killing vector given by

$$\zeta = -k_1 \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x} + \frac{1}{2} y \left[k_1 c_2 - c_1 k_2 + 2(k_1 c_3 - k_3 c_1)\right] \frac{\partial}{\partial y} + \frac{1}{2} z \left[k_1 c_2 - c_1 k_2 - 2(k_1 c_3 - k_3 c_1)\right] \frac{\partial}{\partial z}.$$

Equations (27) and (28) give two relations on the constants c_a and k_a which do not imply necessarily the appearance of more isometries, thus the resulting solutions belong to the tilted Bianchi perfect-fluid models. Nevertheless the study of such solutions are out of the aim of the present work, and will be omitted.

5.2 $n = 1$

Following the arguments given in the subsection 4.2 and looking at (17), in this case we have: $m_1(t) = \dot{T}^2$, $b_i = 0$ and $c_{i1} = \{c_1^2,\ c_2^2,\ c_3^2,\ c_1c_2,\ c_1c_3,\ c_2c_3,\ c_{71},\ c_{81}\}$, where c_{71} and c_{81} are two new arbitrary constants and b_7 and b_8 have been absorbed in M_7 and M_8. The equations containing $\ddot{T}$ are then (17) for $i = 7, 8$ and the t-equation in (13), which read respectively

$$\begin{aligned}
&\left(c_1c_2 - 2c_3^2\right)\dot{T}^2 - \left(\frac{1}{2}c_2 + c_1\right)\ddot{T} + \alpha = c_{71}\dot{T}^2,\\
&\left(c_1 - \frac{1}{2}c_2\right)\left(c_2\dot{T}^2 + \ddot{T}\right) + \beta = c_{81}\dot{T}^2,\\
&c_3\left(c_2\dot{T}^2 + \ddot{T}\right) = \mathrm{K}.
\end{aligned}$$

These expressions together with the fact that the c_a cannot vanish simultaneously imply an equation for $T(t)$ of the form

$$\ddot{T} = a\dot{T}^2 + b,$$

where a and b are constants. Using this equation in the three previous expressions and taking into account that the functions 1 and $\dot{T}^2$ are linearly independent, we get the six following constraints for the constants:

$$c_{71} = c_1 c_2 - 2c_3^2 - \left(c_1 + \frac{1}{2}c_2\right) a, \quad c_{81} = \left(c_1 - \frac{1}{2}c_2\right)(c_2 + a), \quad c_3(c_2 + a) = 0 \tag{29}$$

$$\alpha = \left(\frac{1}{2}c_2 + c_1\right) b, \quad \beta = \left(\frac{1}{2}c_2 - c_1\right) b, \quad \mathrm{K} = c_3 b. \tag{30}$$

The only remaining equation involving t-functions is (22), that reads now $\mathcal{C}_1 \dot{T}^2 + c_{71}c_{81}\dot{T}^4 + B = 0$, implying $\mathcal{C}_1 = B = 0$ and

$$c_{71}c_{81} = 0. \tag{31}$$

The equations (13), (19) and (20) for the x-functions become respectively

$$X_3'' + X_3' X_2' = c_3 b, \qquad \sum_{i=1}^{8} c_{i1} N_i = 0, \qquad N_7 N_8 = 0.$$

From this last relation and (31) it follows that only two different cases may appear (if we neglect the solutions separable in comoving coordinates): (a) $c_{71} \neq 0$ ($\Leftrightarrow \{c_{81} = 0, N_8 = 0, N_7 \neq 0\}$) and (b) $c_{71} = 0$ ($\Leftrightarrow \{c_{81} \neq 0, N_8 \neq 0, N_7 = 0\}$). The final system of equations for both subcases can be written in the following compact normal form once we have defined $\phi \equiv X_2'$, $\psi \equiv X_1' - (1/2)X_2'$ and $\varphi \equiv X_3'$ and after using (30)

$$\varphi' = c_3 b - \phi\varphi, \tag{32}$$

$$\psi' = \left(\frac{1}{2}c_2 + c_1\right) b - \phi\psi + \frac{\epsilon}{\mathcal{K}} \left(c_1\phi + c_2\psi\right)^2, \tag{33}$$

$$\phi' = \phi\left(\frac{1}{2}\phi + 2\psi\right) - 2\varphi^2 - c_2 b - \frac{1}{\mathcal{K}} \left(c_1\phi + c_2\psi - 2c_3\varphi\right)^2, \tag{34}$$

where $\{\epsilon, \mathcal{K}(\neq 0)\} = \{0, c_{71}\}, \{1, c_{81}\}$ for cases (a) or (b) respectively.

In case (a) the solution of the constraints (29) (now $c_{81} = 0$) give two possibilities: (a1) $c_2 + a \neq 0$, implying then $c_2 = 2c_1 \neq 0$, $c_3 = 0$, thus $\mathcal{K} = c_{71} = 2c_1(c_1 - a) \neq 0$, and (a2) $a = -c_2$, which gives $\mathcal{K} = c_{71} = 2c_1c_2 + c_2^2/2 - 2c_3^2 \neq 0$. For case (b), the constraints imply $c_3 = 0$, $a = 2c_1c_2/(c_2 + 2c_1)$ with $c_2 + 2c_1 \neq 0$ (as otherwise it would follow the vanishing of the c_a) and $\mathcal{K} = c_{81} = c_2(c_1 - c_2/2)(c_2 + 4c_1)/(c_2 + 2c_1) \neq 0$. At this point it is interesting to give the conditions (23) explicitly, which read

$$\begin{array}{lll} \text{case (a)} & c_{71} > 0, & c_{71}^2 \dot{T}^2 > \left(c_1\phi + c_2\psi - 2c_3\varphi\right)^2, \\ \text{case (b)} & c_{81} > 0, & c_{81}^2 \dot{T}^2 < \left(c_1\phi + c_2\psi - 2c_3\varphi\right)^2, \end{array}$$

thus, the definition of the $\mathcal{A}_E$-region (where $\rho + p > 0$ holds) gives in general a proper determination of a domain in the manifold and also a condition on the constants, which is invariant under the change $t \leftrightarrow x$. This last feature is important because

these "invariant" conditions allow us to prove some statements about the existence of solutions under some extra conditions (kind of equation of state, for example) that will hold also after changing $n \leftrightarrow q$. Using the coordinate interchange, the previous conditions become $c_{71} > 0$ and $c_{71}^2 X'^2 < (c_1\phi + c_2\psi - 2c_3\varphi)^2$ (now the functions $\{\phi, \psi, \varphi\}$ are redefined with $T_a(t)$ replacing $X_a(x)$) in the case (a) and $c_{81} > 0$ and $c_{81}^2 X'^2 > (c_1\phi + c_2\psi - 2c_3\varphi)^2$ in the case (b).

In the present case, imposing any further restriction such as equations of state may overdetermine the system of equations and give solutions with more isometries. For instance, it can be shown that an equation os state of the form $p = \gamma\rho$ implies $\gamma = 1$ (stiff fluid), even for the comoving solutions, as otherwise a third isometry appears acting on the original Killing orbits, which became plane (i.e. a plane G_3 on S_2).

The system of equations for the different cases has not been explicitly solved in general, but some particular families have been found under some extra restrictions on the constants. In fact, when $b = 0$ in case (a) the system can be completely solved, although the solutions will have $q \leq 4$. Analogously to what has been explained for the case $n = 0$, the values of q for the representative solutions of this case $n = 1$ should be $\{0, 1, 5, 6, 7, 8\}$. We will present now some results concerning solutions with $q = 0, 1$.

5.2.1 $q = 0$

The solutions with $q = 0$ ($X_a'(x) = k_a$) correspond to the singular points $(\varphi_0, \psi_0, \phi_0)$ of the differential system. Let us begin with the case $c_3 \neq 0$ ($\Rightarrow a = -c_2$) (case (a2)), and determine then b from (32). In order to keep $w_1 > 0$ we must avoid $\phi_0 = 0$. Therefore, equations (33) and (34) give

$$\begin{aligned} &\psi_0 = \frac{c_2 + 2c_1}{2c_3}\varphi_0, \\ &(c_3\phi_0 - c_2\varphi_0)\left[\left(2c_1^2 - c_{71}\right)c_3\phi_0 - \left(2c_1^2 + c_{71}\right)c_2\varphi_0\right] = 0, \end{aligned} \tag{35}$$

where (see above for case (a2)) $c_{71} = 2c_1c_2 + c_2^2/2 - 2c_3^2$. These solutions do not admit a third isometry nor a barotropic equation of state in general. In fact, it can be shown that the only barotropic equation of state that these maximally G_2 solutions can have is $\rho = p$, and those correspond to the cases with $c_3\phi_0 - c_2\varphi_0 = 0$. Let us focus our attention into the rest of the cases. From (35), and taking into account that neither $2c_1^2 + c_{71}$ nor c_2 can vanish in order to keep $c_{71} > 0$, we determine φ_0, so finally we have

$$\psi_0 = \left(\frac{c_2 + 2c_1}{2c_2}\right)Q\phi_0, \qquad \varphi_0 = \frac{c_3}{c_2}Q\phi_0, \tag{36}$$

where we have defined $Q \equiv (2c_1^2 - c_{71})/(2c_1^2 + c_{71})$ $(Q < 1)$.

At this stage we can compute $p - \rho$ and realize that it is positive everywhere, so the dominant energy condition cannot hold in the perfect-fluid region. Nevertheless, it is important to remark here that, within our treatment of the problem, solutions not satisfying energy (or other) conditions may still be relevant because these conditions

are *not* invariant under the interchange symmetry $t \leftrightarrow x$. Thus, we must always check the physical properties both for the explicitly obtained solution *as well as* for its partner solution with t and x interchanged. Otherwise, the full method will not be coherent. An illustrative example is given, in fact, by the solutions above, because the new solutions obtained by means of the coordinate interchange are well-behaved. In order to see this, let us perform the change $t \leftrightarrow x$ so that the line-element becomes

$$\widehat{ds}^2 = e^{(2\psi_0+\phi_0)t+2c_1X(x)}\left(-\mathrm{d}t^2+\mathrm{d}x^2\right)+ \\ e^{\phi_0 t+c_2X(x)}\left(e^{2(\varphi_0 t+c_3X(x))}\mathrm{d}y^2+e^{-2(\varphi_0 t+c_3X(x))}\mathrm{d}z^2\right), \qquad (37)$$

where ψ_0 and φ_0 are given by (36) and the function $X(x)$ satisfies the equation

$$X''+c_2X'^2=\frac{Q}{c_2}\,\phi_0^2,$$

from where four cases arise:[4]

$$\begin{aligned}
&\text{(i) } X(x)=\frac{1}{c_2}\ln\cosh\left(\phi_0\sqrt{Q}x\right), && x\in(-\infty,\infty),\\
&\text{(ii) } X(x)=\frac{1}{c_2}\ln\sinh\left(\phi_0\sqrt{Q}x\right), && x\in(0,\infty),\\
&\text{(iii) } X(x)=\frac{1}{c_2}\ln\cos\left(\phi_0\sqrt{-Q}x\right), && \phi_0\sqrt{-Q}x\in(-\pi/2,\pi/2),\\
&\text{(iv) } X(x)=\frac{1}{c_2}\ln(qx), && x\in(0,\infty),
\end{aligned}$$

where Q is positive in the first two cases, negative in the third and $Q=0$ ($c_{71}=2c_1^2$) in the fourth. In this last case, the solutions admit the timelike homothetic Killing vector $\partial/\partial t$ (not parallel to the fluid vector), while for the corresponding '$t \leftrightarrow x$' solutions, the homothetic Killing vector becomes $\partial/\partial x$. The solutions do not have a barotropic equation of state unless $2\psi_0+\phi_0=0$ (implied by $c_1=0$), which gives a further isometry, so we will demand $c_1\neq 0$ in the following. The energy density and the pressure read then

$$\widehat{p}+\widehat{\rho}=e^{-2f_1}c_{71}\left[\left(\psi_0+\frac{1}{2}\phi_0\right)^2\frac{1}{c_1^2}-X'^2\right], \qquad \widehat{p}-\widehat{\rho}=-e^{-2f_1}\frac{2c_{71}}{2c_1^2+c_{71}}\phi_0^2,$$

so in the region $\mathcal{A}_E$ ($\widehat{p}+\widehat{\rho}>0$, and remembering $c_{71}>0$), the dominant energy conditions *are fulfilled*. Note that it can be shown that in the case (i) the condition $\widehat{p}+\widehat{\rho}>0$ is automatically satisfied, so the region $\mathcal{A}_E$ covers the entire spacetime. The quantities involved in the rest of the fluid quantities read

$$\widehat{w}_0=c_{71}\left(\psi_0+\frac{1}{2}\phi_0\right)^2\frac{1}{c_1^2}, \qquad \widehat{w}_1=c_{71}X'^2, \qquad \widehat{\Sigma}=c_{71}\left(\psi_0+\frac{1}{2}\phi_0\right)\frac{1}{c_1}X'.$$

[4]The case $X''=0$ falls into the previous case with $n=q=0$.

The case when $c_3 = 0$ needs also $\phi_0 = 0$ in order to avoid further isometries, and $\epsilon = 1$ (case (b)). After straighforward calculations we find, as in the previous family, that $p - \rho$ is positive everywhere thus violating the dominant energy condition. However, we can use the coordinate interchange symmetry again and luckily we get new solutions which do fulfil the dominant energy conditions in the perfect-fluid region. Finally, the metric is thus given by

$$\widehat{ds}^2 = e^{2\psi_0 t} \cos^{-q_1^2}(q_3 x) \left(-\mathrm{d}t^2 + \mathrm{d}x^2\right) + \cos^{q_2^2}(q_3 x) \left(e^{2\varphi_0 t}\mathrm{d}y^2 + e^{-2\varphi_0 t}\mathrm{d}z^2\right), \qquad (38)$$

where $x \in (-\pi/q_3, \pi/q_3)$, we need to impose $\varphi_0^2 - \psi_0^2 > 0$, and we have defined $q_1^2 \equiv (\varphi_0^2 - \psi_0^2)/2\varphi_0^2$, $q_2^2 \equiv (\varphi_0^2 - \psi_0^2)/(\varphi_0^2 + \psi_0^2)$, and

$$q_3 \equiv 2\varphi_0^2 \sqrt{\frac{2(\varphi_0^2 + \psi_0^2)}{(\varphi_0^2 - \psi_0^2)(3\varphi_0^2 + \psi_0^2)}}.$$

The energy and the pressure are given by

$$\widehat{p} + \widehat{\rho} = \frac{q_2^2}{q_1^2} e^{-2\psi_0 t} \cos^{q_1^2}(q_3 x) \left(2\psi_0^2 \tan^2(q_3 x) - \frac{1}{2} q_1^4 q_3^2\right),$$

$$\widehat{p} - \widehat{\rho} = -q_3^2 q_2^2 e^{-2\psi_0 t} \cos^{q_1^2}(q_3 x) \left(\frac{2\psi_0^2}{\varphi_0^2 + \psi_0^2} \tan^2(q_3 x) + 1\right),$$

from where it is evident that, in the perfect-fluid region $\widehat{p} + \widehat{\rho} > 0$, the dominant energy condition is always satisfied, as claimed previously. There is no barotropic equation of state in general, though. For the rest of the fluid quantities we have

$$\widehat{w_0} = 2\psi_0^2 \frac{q_2^2}{q_1^2} \tan^2(q_3 x), \qquad \widehat{w_1} = \frac{1}{2} q_1^2 q_2^2 q_3^2, \qquad \widehat{\Sigma} = q_2^2 q_3 \psi_0 \tan(q_3 x).$$

These two families of solutions (37) and (38) with $q = 0$ are algebraically general.

5.2.2 $q = 1$

For $q = 1$ we need $X_a(x) = d_a X(x)$ such that $X''(x) \neq 0$, which is also sufficient, because the system of equations (32)–(34) implies necessarily $N_7, N_8 \propto X'^2$. The compatibilization of this system gives some relations on the constants: first, we have that $\epsilon = 0$, so that the solutions must belong to case (a), in order to avoid comoving coordinates. For $b = 0$, the only maximally G_2 solutions without $p = \rho$ equation of state are then given, after redefining the constants, by the following line-element

$$ds^2 = x^{\mu} t^{-2\left(1-\mu^2/q\right)} \left[-\mathrm{d}t^2 + \mathrm{d}x^2 + x^{1-\mu}\left(x^{\nu}\mathrm{d}y^2 + x^{-\nu}\mathrm{d}z^2\right)\right], \qquad (39)$$

where $q \equiv \mu^2 + \nu^2 - 2\mu - 1$, and the corresponding metric after the $t \leftrightarrow x$ interchange. These solutions admit two further conformal Killing vector fields and have no barotropic

equation of state in general. These families were found in [8] (pp.2320), and we refer to this reference for further details.

For $b \neq 0$ we need $c_3 \neq 0 \neq d_3$ as otherwise there would appear a third isometry, so that the solutions must belong to case (a2). Then

$$a = -c_2, \qquad c_2 = -\frac{d_2}{d_3} c_3, \qquad c_1 = d_1 = 0,$$

where d_2 and c_2 cannot vanish so as to have $c_{71} > 0$. The functions $T(t)$ and $X(x)$ are given by $\ddot{T} = d_2 c_3/d_3 \dot{T}^2 + b$ and $X'' = -d_2 X'^2 + c_3 b/d_3$, and the equation of state reads $p = \rho + 2d_2 c_3 b/d_3$. To assure the dominant energy condition we should first impose $b \equiv -d_3\mu^2/(d_2 c_3)$, which turns out to be also sufficient (in the $\mathcal{A}_E$-region). The coordinate interchange $t \leftrightarrow x$ gives no further solutions for the corresponding $\widehat{\mathcal{A}}_E$-regions (which become, in fact, identical to $\mathcal{A}_E$). Two different families of solutions appear. The first is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \cos^{1+2\nu}(\mu x)\cosh^{1-2\nu}(\mu t)\mathrm{d}y^2 + \cos^{1-2\nu}(\mu x)\cosh^{1+2\nu}(\mu t)\mathrm{d}z^2, \quad (40)$$

where μ and $\nu \geq 0$ are constants. The conditions for the $\mathcal{A}_E$ region give $\nu \leq 1/2$ and

$$\tanh^2(\mu t) > \tan^2(\mu x),$$

so that we choose $t \geq 0$ and $x \in (-\pi/4\mu, \pi/4\mu)$. The equation of state and the fluid quantities are given by

$$\rho = \mu^2\left[\left(\frac{1}{4} - \nu^2\right)\left(\tanh^2(\mu t) - \tan^2(\mu x)\right) + 1\right], \qquad p = \rho - 2\mu^2,$$

$$u^0 = \frac{\tanh(\mu t)}{\sqrt{\tanh^2(\mu t) - \tan^2(\mu x)}}, \qquad u^1 = -\frac{\tan(\mu x)}{\sqrt{\tanh^2(\mu t) - \tan^2(\mu x)}},$$

$$\mathrm{a}_1 = -\mu(u^0u^1)^2\tan^{-1}(\mu x)(\tanh^2(\mu t) - \tan^2(\mu x) - 2), \qquad \mathrm{a}_0 = \frac{\tan(\mu x)}{\tanh(\mu t)}\mathrm{a}_1,$$

$$\theta = \mu\frac{(u^0)^2 + (u^1)^2}{\sqrt{\tanh^2(\mu t) - \tan^2(\mu x)}}\left(\tanh^2(\mu t) - \tan^2(\mu x) - 1\right),$$

$$\sigma_{00} = \left(\frac{\tan(\mu x)}{\tanh(\mu t)}\right)^2\sigma_{11}, \qquad \sigma_{01} = \frac{\tan(\mu x)}{\tanh(\mu t)}\sigma_{11}$$

$$\sigma_{11} = -\frac{\mu\tanh^2(\mu t)\left[(u^0)^2 + (u^1)^2\right]}{3\left(\tanh^2(\mu t) - \tan^2(\mu x)\right)^{3/2}}\left(\tanh^2(\mu t) - \tan^2(\mu x) + 2\right)$$

$$\sigma_{22} = -\mu\nu\sqrt{\tanh^2(\mu t) - \tan^2(\mu x)} + \frac{1}{2(u^0)^2}\sigma_{11}$$

$$\sigma_{33} = 2\mu\nu\sqrt{\tanh^2(\mu t) - \tan^2(\mu x)} + \sigma_{22}.$$

The non-vanishing components of the Weyl tensor are

$$\Psi_0 = \Psi_4 = 2\nu\mu^2, \qquad \Psi_2 = \frac{1}{2}\mu^2 - \frac{1}{3}\rho,$$

thus the Petrov type is I for generic points.

Notice that this family of perfect-fluid solutions do not present any curvature singularity. However, this fact may be not relevant because, as explained above, the perfect-fluid $\mathcal{A}$-region is extendible through the spacelike hypersurface $\tanh^2(\mu t) - \tan^2(\mu x) = 0$. A particular extension is in fact given by the line-element (40) itself when taking every possible value of t and the range $x \in (-\pi/\mu,\, \pi/\mu)$. This particular extension is then singular, as can be immediately checked from the above expressions.

This solution (40) is in fact a good example of the power of using non-comoving coordinates. As explained in section 2, every diagonal G_2 solution can be written in comoving coordinates. Thus, we can ask ourselves, how does solution (40) look like in comoving coordinates? In this case the change of separable coordinates to comoving coordinates $\{t', x'\}$ can be performed explicitly and the line-element (40) in the $\mathcal{A}$-region becomes

$$ds^2 = \frac{1}{2}\left(t'^2 + 1 + \sqrt{f}\right)\left\{\frac{1}{\mu^2\sqrt{f}}\left(-\frac{1}{t'^2}\mathrm{d}t'^2 + \frac{t'}{t'^2 + 1 + \sqrt{f}}\mathrm{d}x'^2\right)\right.$$
$$\left. + t'^{\,-(2\nu+1)}\mathrm{d}y^2 + t'^{\,(2\nu-1)}\mathrm{d}y^2\right\},$$

where $f(t', x') \equiv (t'^2 - 1)^2 - x'^2$ and the ranges of t' and x' are restricted to $t' > 0$ and $f(t', x') > 0$. As we can check, it may be difficult to find such solution if we use comoving coordinates. Compare with its original form (40), which is really *simple*.

The second family with $q = 1$ can be obtained by simply replacing the $\cosh(\mu t)$ functions by $\sinh(\mu t)$ everywhere (so $\tanh(\mu t) \to \coth(\mu t)$). In this case the range for t is restricted to $t > 0$ because there is an initial spacelike singularity at $t = 0$.

5.3 $n = 2$

Despite the fact that now there are two different subcases, we begin giving a common feature for them: we must put $c_3 = 0$ in order to have $\ddot{T}$ linearly independent from 1 and $\dot{T}^2$ (see (13)).

In the subcase (i) we take $\{1, \dot{T}^2, M_8\}$ as linearly independent functions. To this end, due to the definition of M_8, we must also demand that $c_1 - c_2/2 \neq 0$. We impose then

$$M_7 = c_{71}\dot{T}^2 + c_{72}M_8,$$

which must be an identity, i.e. multiplying this expression by $c_1 - c_2/2$ we get a linear relation between the three linearly independent functions and thus the constant coefficients must vanish, providing the following constraints on the contants

$$c_{71} = c_2\left(2c_1 + \frac{1}{2}c_2\right), \qquad c_{72} = \frac{c_2 + 2c_1}{c_2 - 2c_1}, \qquad \alpha = c_{72}\beta.$$

The rest of the c_{iA} constants are given by $c_{L1} = \{c_1^2,\, c_2^2,\, 0,\, c_1c_2,\, 0,\, 0\}$, $c_{L2} = 0$, $c_{81} = 0, c_{82} = 1$, while $b_i = 0$. Regarding the t-functions, it only remains the relation (22), which gives the equation for $T(t)$:

$$\ddot{T}^2(2c_1 + c_2) + 2\ddot{T}(\mathcal{C}_2 - 2\beta c_{72}) + \ddot{T}\dot{T}^2 c_2^2 + 2\dot{T}^2\left(\frac{\beta c_2^2 - 2\mathcal{C}_1}{2c_1 - c_2} - c_2\mathcal{C}_2\right) - 2\dot{T}^4 c_2^2 c_1 = 4\frac{(\beta^2 c_{72} + \beta\mathcal{C}_2 + B)}{2c_1 - c_2}.$$

The system for the x-functions (13), (45) and (19) is given by

$$\begin{aligned}
&X_3'' + X_3'X_2' = 0,\\
&N_7N_8 = B,\\
&\sum_{L=1}^{6} c_{L1}N_L + c_{71}N_7 = \mathcal{C}_1,\\
&c_{72}N_7 + N_8 = \mathcal{C}_2.
\end{aligned}$$

This system contains four equations for three unknowns. Thus, taking into account that we must impose $X_3' \not\equiv 0$ to avoid a third isometry, the compatibilization of the system gives raise to a particular family of solutions, apart from another solutions which fall into the previous cases $n = 0, 1$. After redefining some constants, the line element reads

$$\mathrm{d}s^2 = e^{2\mu x + 2c_1T(t)}\left(-\mathrm{d}t^2 + \mathrm{d}x^2\right) + e^{c_2T(t)}\left(e^{2\nu x}\mathrm{d}y^2 + e^{-2\nu x}\mathrm{d}z^2\right), \tag{41}$$

where μ and ν are constants, $c_2 \neq 0$, $c_2 + 2c_1 \neq 0$ and $T(t)$ satisfies

$$\ddot{T} = \frac{1}{2(c_2 + 2c_1)}\left(2\epsilon\sqrt{\Delta} - 4c_{72}\nu^2 - c_2^2\dot{T}^2\right),$$

where $\epsilon^2 = 1$ and $\Delta = (c_{71}\dot{T}^2 + 2c_{72}\nu^2)^2 + 4\mu^2c_2^2c_{72}\dot{T}^2$. This family does not contain a barotropic equation of state except for cases with more isometries. The pressure, the energy density and the rest of the variables involved in the fluid quantities are given by

$$\begin{aligned}
&\rho = e^{-2(\mu x + c_1T(t))}\frac{c_2}{c_2 + 2c_1}\left(c_{71}\dot{T}^2 - \frac{(2c_1 - c_2)}{2c_2}\epsilon\sqrt{\Delta}\right),\\
&p + \rho = e^{-2(\mu x + c_1T(t))}\frac{c_2}{c_2 + 2c_1}\left(c_{71}\dot{T}^2 - 2c_{72}\nu^2 - 2\frac{c_1}{c_2}\epsilon\sqrt{\Delta}\right),\ \ \Sigma = c_2\mu\dot{T},\\
&w_0 = \frac{1}{2}\left(c_{71}\dot{T}^2 + 2c_{72}\nu^2 - \epsilon\sqrt{\Delta}\right),\ \ w_1 = -\frac{1}{2c_{72}}\left(c_{71}\dot{T}^2 + 2c_{72}\nu^2 + \epsilon\sqrt{\Delta}\right),
\end{aligned}$$

from where it is straightforward to find the pefect-fluid quantities using the expressions given in the subsection 4.5, and realize that the region defined by $\Delta = 0$ does not

represent any physical singularity but a focal zone for the fluid congruence. The non-zero components of the Weyl tensor read

$$\psi_0+\psi_4=2e^{-2f_1}\mu\nu,\ \psi_0-\psi_4=-e^{-2f_1}\nu(c_2-2c_1)\dot{T},\ \psi_2=\frac{1}{12}e^{-2f_1}\left[(c_2-2c_1)\ddot{T}-4\nu^2\right],$$

where $f_1=\mu x+c_1T(t)$, so the Petrov type is I.

Although the $\mathcal{A}_E$ region is defined in principle by two inequalities involving the function $T(t)$ (i.e. $\rho+p>0$, $w_0>0$), it can be shown that after imposing the former, it suffices the evaluation of the latter on $\mathcal{F}$ (i.e. in $\rho+p=0$), giving then only a condition on the constants involved. This happens because Σ never vanishes at $\mathcal{F}$.

For the subcase (ii) we take $\{1,\dot{T}^2,M_7\}$ to be three linearly independent functions and impose the relation

$$M_8=c_{81}\dot{T}^2.$$

Following the same procedure as in the previous subcase, we must have now $c_2+2c_1\neq 0$, $c_2-2c_1=0$, $\beta=0$, and $c_{81}=0$, thus in fact, $M_8=0$ identically. Therefore, equation (22) has the form

$$\mathcal{C}_1\dot{T}^2+\mathcal{C}_2M_7+B=0,$$

from where it follows that $\mathcal{C}_1=\mathcal{C}_2=B=0$. The constants c_{L1} and c_{L2} are the same as in the subcase (i) and also $c_{71}=0$, $c_{72}=1$. The equation (19) for $A=2$ reads then

$$\sum_{i=1}^{8}c_{i2}N_i=N_7=0,$$

thus $w_1=0$ and therefore the possible solutions belonging to this subcase are always separable in comoving coordinates.

To sum up this subsection, in $n=2$ there is only a family of solutions. Its line element is given by (41) and has no barotropic equation of state.

6 Concluding remarks

The assumption of separability of the metric functions in non-comoving coordinates has been shown to be a valuable tool for the obtaining of inhomogeneous exact solutions. By exploiting the coordinate interchange symmetry explained in section 2, a purely mathematical classification is put forward and provides a systematic way of getting new diagonal G_2 on S_2 solutions. This has been explicitly used in the present work to construct several new solutions by means of the analysis of the simplest case (m=1). It arises a relationship between the different cases of the classification and some physical properties, such as the existence of a barotropic equation of state and its explicit form. Furthermore, the use of non-comoving coordinates shows explicitly that the space-times obtained by the imposition of a particular Segré type of the energy-momentum tensor (for instance, perfect fluid) may be extendible in general and with a varying algebraic type through the extension.

A The Einstein tensor and the functions $M_i(t)$ and $N_i(x)$

Here we give the explicit expressions for the Einstein tensor for the metric given in (12) computed in the natural orthonormal co-basis (4), and the functions $M_i(t)$ and $N_i(x)$ ($i = 1 \ldots 8$) in terms of the metric functions defined upon equations (14) and (15). The non-zero components of Einstein tensor for (12) in the frame (4) are:

$$
\begin{aligned}
S_{00} &= e^{-2f_1}\left[\dot{T}_1\dot{T}_2 + \frac{1}{4}\dot{T}_2^{\,2} - \dot{T}_3^{\,2} - X_2'' - \frac{3}{4}X_2'^2 - X_3'^2 + X_1'X_2'\right],\\
S_{11} &= e^{-2f_1}\left[X_1'X_2' + \frac{1}{4}X_2'^2 - X_3'^2 - \ddot{T}_2 - \frac{3}{4}\dot{T}_2^{\,2} - \dot{T}_3^{\,2} + \dot{T}_1\dot{T}_2\right],\\
S_{01} &= e^{-2f_1}\left[\dot{T}_1X_2' + \dot{T}_2\left(X_1' - \frac{1}{2}X_2'\right) - 2\dot{T}_3X_3'\right],\\
S_{22} &= e^{-2f_1}\left[X_1'' + \frac{1}{2}X_2'' + \frac{1}{4}X_2'^2 - X_2'X_3' - X_3'' + X_3'^2\right.\\
&\qquad\left. -\ddot{T}_1 - \frac{1}{2}\ddot{T}_2 - \frac{1}{4}\dot{T}_2^{\,2} + \dot{T}_2\dot{T}_3 + \ddot{T}_3 - \dot{T}_3^{\,2}\right],\\
S_{33} &= e^{-2f_1}\left[X_1'' + \frac{1}{2}X_2'' + \frac{1}{4}X_2'^2 + X_2'X_3' + X_3'' + X_3'^2\right.\\
&\qquad\left. -\ddot{T}_1 - \frac{1}{2}\ddot{T}_2 - \frac{1}{4}\dot{T}_2^{\,2} - \dot{T}_2\dot{T}_3 - \ddot{T}_3 - \dot{T}_3^{\,2}\right].
\end{aligned}
$$

The equation (13) is now used to eliminate $\ddot{T}_3$ and X_3'' in what follows. There is some freedom in defining the M and N functions, and we could have chosen the definitions (in particular for $M_L(t)$ and $N_L(x)$ ($L = 1 \ldots 6$)) keeping the $x \leftrightarrow t$ symmetry; but, as we deal with the t-functions, we define $M_L(t)$ as simpler as possible. From S_{01}^2 we define $M_L(t)$ and $N_L(x)$ as

$$M_1 = \dot{T}_1^{\,2},\ M_2 = \dot{T}_2^{\,2},\ M_3 = \dot{T}_3^{\,2},\ M_4 = \dot{T}_1\dot{T}_2,\ M_5 = \dot{T}_1\dot{T}_3,\ M_6 = \dot{T}_2\dot{T}_3, \tag{42}$$

$$
\begin{aligned}
N_1 = -X_2'^2, \qquad & N_2 = -\left(X_1' - \tfrac{1}{2}X_2'\right)^2, \qquad N_3 = -4X_3'^2,\\
N_4 = -2X_2'\left(X_1' - \tfrac{1}{2}X_2'\right), \qquad & N_5 = 4X_2'X_3', \qquad N_6 = 4X_3'\left(X_1' - \tfrac{1}{2}X_2'\right).
\end{aligned}
\tag{43}
$$

Similarly, from the right hand side of (6), and using the explicit expressions for the two factors, we can define $M_7(t)$, $M_8(t)$, $N_7(x)$ and $N_8(x)$ by means of

$$
\begin{aligned}
S_{00} + S_{22} &= e^{-2f_1}\left(M_7(t) + N_8(x)\right),\\
S_{11} - S_{22} &= e^{-2f_1}\left(M_8(t) + N_7(x)\right),
\end{aligned}
$$

and such that (15) holds. Thus,

$$M_7 = \dot{T}_1\dot{T}_2 - \tfrac{1}{2}\ddot{T}_2 - 2\dot{T}_3^{\,2} - \ddot{T}_1 + \alpha,\quad M_8 = \dot{T}_1\dot{T}_2 - \tfrac{1}{2}\ddot{T}_2 - \tfrac{1}{2}\dot{T}_2^{\,2} + \ddot{T}_1 + \beta, \tag{44}$$

$$N_7 = X_1'X_2' - \tfrac{1}{2}X_2'' - 2X_3'^2 - X_1'' - \beta,\quad N_8 = X_1'X_2' - \tfrac{1}{2}X_2'' - \tfrac{1}{2}X_2'^2 + X_1'' - \alpha, \tag{45}$$

where α and β are two arbitrary constants. With the help of these constants we can always set, for instance, $M_7(t)$+const.$\rightarrow M_7(t)$, that is, we can absorve the constants added to M_7, M_8 into themselves. This feature is very useful to simplify the expressions and the equations for $N_i(x)$.

B Some useful results

Here we give some lemmas concerning sets of linearly independent functions which will be useful for relating the integers m and n defined in the section 4. The first two lemmas relate a set of functions $\{f_i\}$ $(i = 1 \ldots r)$, from (an interval of) $\mathbb{R}$ to $\mathbb{R}$, with its derivatives (denoted by a dot). The rest give some relations of the same kind between the sets $\{f_i\}$ and $\{f_i f_j\}$ (the combinations of products of two fuctions from $\{f_i\}$). We will present an example of each type of proof to give an outline of the procedures involved.

Lemma 1: *Let $\{f_i\}$ be a set of r linearly independent C^1 functions. Then, there are at least $r-1$ linearly independent functions among the set $\{\dot{f}_i\}$.*

Proof. Suppose that there were only $r-2$ linearly independent functions among $\{\dot{f}_i\}$. This would mean that we have two independent relations between them

$$\sum_{i=1}^{r} a_i \dot{f}_i = 0, \qquad \sum_{i=1}^{r} b_i \dot{f}_i = 0,$$

where a_i and b_i are constants (or functions depending on another independent variable). Integrating these relations we would get

$$\sum_{i=1}^{r} a_i f_i = A \neq 0, \qquad \sum_{i=1}^{r} b_i f_i = B \neq 0$$

where A and B cannot vanish. Therefore

$$\sum_{i=1}^{r} (Ba_i - Ab_i) f_i = 0,$$

and thus $Ba_i - Ab_i = 0$ because $\{f_i\}$ are r linearly independent functions. But then, the two relations in (B) would be linearly dependent, which is a contradiction.

Lemma 2: *$\{\dot{f}_i\}$ is a set of r linearly independent functions if and only if $\{f_i, 1\}$ is a set of $r+1$ linearly independent functions.*

Lemma 3: *Let f_1 and f_2 be two linearly independent functions such that the intersection of their supports is non empty. Then, the set $\{f_i f_j\} = \{f_1^2, f_2^2, f_1 f_2\}$ contains three linearly independent functions.*

Let us note that the condition on the supports avoids $f_1 f_2 \equiv 0$ (in all the interval of definition of $\{f_i\}$).
Proof. Suppose, on the contrary, that there existed a relation $af_1^2 + bf_2^2 + cf_1 f_2 = 0$. Dividing this relation by f_1^2 (in its support), we would get a polynomial for f_2/f_1. This would imply that a, b, and c are all equal to zero, in order to avoid the proportionality of f_1 and f_2.

Theorem 1 *Let f_1, f_2, and f_3 be three linearly independent functions such that the intersection of any two of their supports is non empty. Then, the set $\{f_i f_j\}$ contains at least five linearly independent functions.*

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